

Polynomials on Parabolic Manifolds

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ABSTRACT. A Stein manifold X is called S -parabolic if it possesses a plurisubharmonic exhaustion function ρ that is maximal outside a compact subset of X . In analogy with $(\mathbb{C}^n, \ln|z|)$, one defines the space of polynomials on a S -parabolic manifold (X, ρ) as the set of all analytic functions with polynomial growth with respect to ρ . In this work, which is, in a sense continuation of [7], we will primarily study polynomials on S -parabolic Stein manifolds. In Section 2, we review different notions of parabolicity for Stein manifolds, look at some examples and go over the connections between parabolicity of a Stein manifold X and certain linear topological properties of the Fréchet space of global analytic functions on X . In Section 3 we consider Lelong classes, associated Green functions and introduce the class of polynomials in S -parabolic manifolds. In Section 4 we construct an example of a S -parabolic manifold, with no nontrivial polynomials. This example leads us to divide S -parabolic manifolds into two groups as the ones whose class of polynomials is dense in the corresponding space of analytic functions and the ones whose class of polynomials is not so rich. In this way we introduce a new notion of regularity for S -parabolic manifolds. In the final section we investigate linear topological properties of regular S -parabolic Stein manifolds and show in particular that the space of analytic functions on such manifolds have a basis consisting of polynomials. We also give a criterion for closed submanifolds of a regular S -parabolic to be regular S -parabolic, in terms of existence of tame extension operators for the spaces of analytic functions defined on these submanifolds.

1. Introduction

In the classical theory of Riemann surfaces one calls a Riemann surface parabolic, in case every bounded (from above) subharmonic (sh) function on X reduces to a constant. Several authors introduced analogs of these notions for general complex manifolds of arbitrary dimension in different ways; in terms of triviality (parabolic type) and non-triviality (hyperbolic type) of the Kobayashi or Caratheodory metrics, in terms of plurisubharmonic (psh) functions, etc. In this paper we will follow the one dimensional tradition and call a complex manifold parabolic in case every bounded from above plurisubharmonic function on it reduces to a constant.

On the other hand, Stoll, Griffiths, King, et al. in their work on Nevanlinna's value distribution theory in higher dimensions, introduced notions of "*parabolicity*" in several complex variables by requiring the existence of *special plurisubharmonic*

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(*psh*) *exhaustion functions*. Following Stoll [33], we will call an n -dimensional complex manifold X , S -*parabolic* in case there is a plurisubharmonic function ρ on X with the properties:

- a) $\{z \in X : \rho(z) < C\} \subset\subset X, \forall C \in \mathbb{R}$ (i.e. ρ is exhaustive),
 - b) the Monge - Ampère operator $(dd^c \rho)^n$ is zero off a compact $K \subset\subset X$.
- That is ρ is maximal plurisubharmonic function outside K .

If in addition we can choose ρ to be continuous then we will say that X is S^* -parabolic. Special exhaustion functions with certain regularity properties play a key role in the Nevanlinna's value distribution theory of holomorphic maps $f : X \rightarrow P^m$, where P^m is the m -dimensional projective manifold (see. [13], [23], [29], [32], [33]).

On the other hand, for manifolds which have a special exhaustion function one can define extremal Green functions as in the classical case and apply pluripotential theory techniques to obtain analogs of some classical results (see [26], [38], [39]) and section 3 below.

Most of the previous papers on the subject required additional smoothness conditions for the special exhaustion functions. Note that we only distinguish the cases when the special exhaustion function is continuous or just plurisubharmonic. Also note that without the maximality condition b), an exhaustion function $\rho(z) \in C(X) \cap psh(X)$ always exists for any Stein manifold X . This follows from the fact that any Stein manifold X , $\dim(X) = n$, can be properly embedded in \mathbb{C}_W^{2n+1} , hence one can take for ρ the restriction of $\ln |w|$ to X .

In this paper, we will primarily study polynomials in S -parabolic Stein manifolds. Polynomials in S^* -parabolic manifolds were introduced by A. Zeriahi in [38]. However, his investigations were more focused on polynomials on affine algebraic varieties. In analogy with $(\mathbb{C}^n, \ln |z|)$ one defines polynomials in a S -parabolic manifold (X, ρ) as the set of all analytic functions with polynomial growth with respect to ρ .

The organization of the paper is as follows: In 2, we review different notions of parabolicity for Stein manifolds, look at some examples and go over the connections between parabolicity of a Stein manifold X and certain linear topological properties of the Fréchet space of global analytic functions on X . In 3 we consider Lelong classes, associated Green functions and introduce the class of polynomials in S -parabolic manifolds. These two sections are written in a survey style. In 4 we construct an example of a S^* -manifold, with no nontrivial polynomials. This example leads us to divide S -parabolic manifolds into two groups as the ones whose class of polynomials is dense in the corresponding space of analytic functions and the ones whose class of polynomials is not so rich. In this way we introduce a new notion of regularity for S -parabolic manifolds. In the final section we investigate linear topological properties of regular S -parabolic Stein manifolds and show in particular that the space of analytic functions on such manifolds have a basis consisting of polynomials. In this section we also give a criterion for closed submanifolds of a regular S^* -parabolic to be regular S^* -parabolic, in terms of existence of tame extension operators for the spaces of analytic functions defined on these submanifolds.

2. Parabolic manifolds

In this section we will review notions of parabolicity for Stein manifolds, look at some examples and go over the relation between parabolicity of a Stein manifold

X and certain linear topological properties of the Fréchet space of global analytic functions on X .

DEFINITION 2.1. A Stein manifold X is called parabolic, in case it does not possess a non-constant bounded above plurisubharmonic function.

Thus, parabolicity of X is equivalent to the following: if $u(z) \in psh(X)$ and $u(z) < C$, then $u(z) \equiv \text{const}$ on X .

It is very convenient to describe parabolicity in term of well-known \mathcal{P} -measures of pluripotential theory [7], [24]. Let our Stein manifold X be properly imbedded in \mathbb{C}_w^{2n+1} , $n = \dim X$, and denote by $\sigma(z)$ the restriction of $\ln|w|$ to X . Then $\sigma(z) \in C(X) \cap psh(X)$ and $\{\sigma(z) < C\} \subset\subset X, \forall C \in \mathbb{R}$. We assume 0 is not in X and that, $\min \sigma(z) < 0$. We consider σ -balls $B_R = \{z \in X : \sigma(z) < \ln R\}$ and as usual, define the class $\mathcal{U}(\overline{B}_1, B_R) = \{u \in psh(B_R) : u|_{\overline{B}_1} < -1, u|_{B_R} < 0\}$. Then the function

$$\omega(z, \overline{B}_1, B_R) = \sup\{u(z) : u \in \mathcal{U}(\overline{B}_1, B_R)\}$$

is called as \mathcal{P} -measure of the \overline{B}_1 with respect to the domain B_R . \mathcal{P} -measure $\omega(z, \overline{B}_1, B_R)$ is plurisubharmonic in B_R , is equal to -1 on \overline{B}_1 and tends to 0 for $z \rightarrow \partial B_R$. Moreover, it is maximal, i.e. $(dd^c \omega)^n = \text{in } B_R \setminus \overline{B}_1$. Since $\omega(z, \overline{B}_1, B_R)$ decreases with $R \nearrow \infty$, and the limiting function satisfies:

$$\omega(z, \overline{B}_1) = \lim_{R \rightarrow \infty} \omega(z, \overline{B}_1, B_R) \in psh(X), \quad \omega(z, \overline{B}_1)|_{\overline{B}_1} \equiv -1, \quad \omega(z, \overline{B}_1) < 0 \quad \forall z \in X.$$

The proposition below, while not difficult to prove, is sometimes very useful.

PROPOSITION 2.2. *The Stein manifold X is parabolic if and only if $\omega(z, \overline{B}_1)$ is trivial, i.e. $\omega(z, \overline{B}_1) \equiv -1$.*

We note, that triviality of $\omega(z, \overline{B}_1)$ does not depend upon \overline{B}_1 ; one can take instead of \overline{B}_1 , any closed ball \overline{B}_r or even, any pluriregular compact set $E \subset X$ (see [7]).

DEFINITION 2.3. A Stein manifold X is called S -parabolic, if there exist exhaustion function $\rho(z) \in psh(X)$ that is maximal outside a compact subset of X . If in addition we can choose $\rho(z)$ to be continuous then we will say that X is S^* -parabolic.

A plurisubharmonic exhaustion function that is maximal outside a compact subset will be referred to as special plurisubharmonic exhaustion. We will tacitly assume, unless otherwise stated that special exhaustion functions are maximal on the sets where they are strictly positive. It is not difficult to see that S -parabolic manifolds are parabolic.

In fact, since the special exhaustion function $\rho(z)$ of a S -parabolic manifold (X, ρ) is maximal off some compact $K \subset\subset X$ we can choose a positive r , so that $B_r = \{\rho(z) < \ln r\}$ contains K . For $R > r$ the \mathcal{P} -measure can be calculated as;

$$\omega(z, \overline{B}_1, B_R) = \left\{ -1, \frac{\rho(z) - R}{R - r} \right\},$$

From here it follows, that $\lim_{R \rightarrow \infty} \omega(z, \overline{B}_1, B_R) \equiv -1$.

For open Riemann surfaces the notions of S -parabolicity, S^* -parabolicity and parabolicity coincide. This is a consequence of the existence of Evans-Selberg potentials (subharmonic exhaustion functions that are harmonic outside a given point)

on parabolic Riemann surfaces [28]. Authors do not know any prove of the following important problems in the multidimensional case $n = \dim X > 1$.

PROBLEM 2.4. Do the notions of S -parabolicity and S^* -parabolicity coincides for the Stein manifolds of arbitrary dimension?

PROBLEM 2.5. Do the notions of parabolicity and S -parabolicity coincide for the Stein manifolds of arbitrary dimension?

The prime example of an S^* -parabolic manifold is of course \mathbb{C}^n , with the special exhaustion function $\ln^+ |z|$. Algebraic affine manifolds, with their canonical special exhaustion functions as described in [13] also forms an important class of S^* -parabolic manifolds.

Another set of indicative examples could be obtained by considering closed pluripolar subsets in \mathbb{C}^n , whose complements are pseudoconvex. Such sets are called “*analytic multifunctions*” by some authors. They are studied extensively and are extremely important in approximation theory, in the theory of analytic continuation and in the description of polynomial convex hulls (see [1, 8, 17, 18, 27, 30, 31] and others). It is clear, that these sets are removable for the class of bounded plurisubharmonic functions defined on their complements. Hence their complements are parabolic Stein manifolds. We would like to state the following special case of Problem 2.5 above, with the hope that it will be more tractable:

PROBLEM 2.6. Let A be an analytic multifunction in \mathbb{C}^n . Is $X = \mathbb{C}^n \setminus A$, S -parabolic?

In classical case, $n = 1$, every closed polar set $A \subset \mathbb{C}$ is an analytic multifunction. As was remarked above, in this case $\mathbb{C}^n \setminus A$ is S^* -parabolic. On the other hand if $A = \{p(z) = 0\} \subset \mathbb{C}^n$ is an algebraic set, where p is a polynomial, assuming that $0 \notin A$, it is not difficult to see that the function

$$\rho(z) = -\frac{1}{\deg p} \ln |p(z)| + 2 \ln |z|$$

gives rise to a special exhaustion function for $\mathbb{C}^n \setminus A$. More generally we have:

THEOREM 2.7 ([7]). *Let*

$$A = \{z = ({}'z, z_n) = (z_1, z_2, \dots, z_n) \in \mathbb{C}^n : z_n^k + f_1({}'z)z_n^{k-1} + \dots + f_k({}'z) = 0\}$$

be a Weierstrass polynomial (algebraic) set, where $f_j \in O(\mathbb{C}^{n-1})$ are entire functions, $j = 1, 2, \dots, k$, $k > 1$. Then $X = \mathbb{C}^n \setminus A$ is S^ -parabolic. Moreover, the function $\rho(z) = -\ln |F(z)| + \ln (|z|'^2 + |F(z) - 1|^2)$ is a plurisubharmonic exhaustion function for X , that is maximal outside a compact subset of X .*

For more examples of parabolic manifolds we refer the reader to [7].

It turns out that the parabolicity of a Stein manifold X and certain linear topological properties of the Fréchet space of analytic functions on X are connected. We will end this section by reviewing some results obtained in this context.

As usual, the topology on the space of analytic functions on a complex manifold X , $O(X)$ is the topology of uniform convergence on compact subsets of X , which makes $O(X)$ a nuclear Fréchet space. We start by recalling the DN condition of Vogt from the structure theory of Fréchet spaces;

DEFINITION 2.8. A Fréchet space Y has the *property DN* in case for a system $(\|\cdot\|_k)$ of seminorms generating the topology of Y one has:

$$\exists k_0 \text{ such that } \forall p \exists q, C > 0 : \|x\|_p \leq C \|x\|_{k_0}^{\frac{1}{2}} \|x\|_q^{\frac{1}{2}}, \forall x \in Y$$

This condition does not depend on the choice of generating seminorms. For this and related linear topological invariants we refer the reader to the book ([15]).

The first result we will state is an adaptation of a result from ([2]) part of which were proved by D.Vogt, V.Zaharyuta, and the first author independently.

THEOREM 2.9. *For a Stein manifold X of dimension n , the following conditions are equivalent:*

1. X is parabolic
2. $O(X)$ has the property DN
3. $O(X)$ is isomorphic as Fréchet spaces to $O(\mathbb{C}^n)$.

Mitiagin and Henkin, in their seminal paper ([16]) initiated a program which they called "linearization of the basic theorems of complex analysis". One of the problems they considered (in connection with Remmert's theorem) was the possibility of finding continuous linear right inverse operators to the restriction operator for analytic functions defined on closed complex submanifolds of \mathbb{C}^N . In other words for a closed complex submanifold V of some \mathbb{C}^N , denoting by R the restriction operator from $O(\mathbb{C}^N)$ onto $O(V)$ the query was to find a continuous linear (extension) operator $E : O(V) \rightarrow O(\mathbb{C}^N)$ such that $R \circ E = \text{Identity}$ on $O(V)$. Mitiagin and Henkin stated (Proposition 6.5 [16]) that this was possible in case $O(V)$ is isomorphic to $O(\mathbb{C}^n)$, $n = \dim V$. A complete answer to this query was given by Vogt [34] (see also [35], [36]), which in our terminology reads as follows:

THEOREM 2.10. *A Stein manifold is parabolic if and only if whenever it is embedded into a Stein manifold as a closed submanifold, it admits a continuous linear extension operator.*

We now wish to pass to a more refined category of Fréchet spaces. Recall that a graded Fréchet space is a tuple $(Y, \|\cdot\|_s)$, where Y is a Fréchet space and $(\|\cdot\|_s)$ is a fixed system of seminorms on Y defining the topology. The morphisms in this category are tame linear operators.

DEFINITION 2.11. A continuous linear operator T between two graded Fréchet spaces $(Y, \|\cdot\|_s)$ and $(Z, \|\cdot\|_k)$ is said to be *tame* in case:

$$\exists A > 0 \forall k \exists C > 0 : |T(x)|_k \leq C \|x\|_{k+A}.$$

Two graded Fréchet spaces are called tamely isomorphic in case there is a one to one tame linear operator from one onto the other whose inverse is also tame.

On a Stein manifold X , each exhaustion $(K_s)_{s=1}^\infty$ of holomorphically convex compact sets with $K_s \subset\subset \text{int} K_{s+1}$, $s = 1, 2, \dots$, induces a grading $\{\|\cdot\|_{K_s}\}$ on $O(X)$ by considering the sup norms on these compact sets.

THEOREM 2.12. ([7]) *A Stein manifold of dimension n is S^* -parabolic if and only if there exists an exhaustion $(K_s)_{s=1}^\infty$ of X such that the graded spaces $(O(X), \|\cdot\|_{K_s})$ and $(O(\mathbb{C}^n), \|\cdot\|_{P_s})$ are tamely isomorphic, where $P_s = \{z \in \mathbb{C}^n : \|z\| \leq e^s\}$, $s = 1, 2, \dots$.*

This result displays the similarities between function theories on S^* -parabolic manifolds and the complex Euclidean spaces, however finding tame isomorphisms to the space of entire functions may not be an easy task. On the other hand graded Fréchet spaces tamely isomorphic to *infinite type power series spaces* were studied by various authors (see for example, [20]) and linear topological conditions that ensure the existence of such isomorphisms were obtained. Recall that for an exponent sequence $\alpha = (\alpha_m)_m$; $\alpha_m \uparrow \infty$, the power series space of infinite type is the graded Fréchet space

$$\Lambda_\infty(\alpha) = \left\{ \xi = (\xi_m)_m : |\xi|_k \doteq \sum_{m=1}^{\infty} |\xi_m| e^{k\alpha_m} < \infty, \ k = 1, 2, \dots \right\}$$

equipped with the grading $(|\cdot|_k)_{k=1}^\infty$.

THEOREM 2.13. ([7]) *A Stein manifold X of dimension n is S^* -parabolic in case there exists an exhaustion $(K_s)_{s=1}^\infty$ of X such that $(O(X), \|\cdot\|_{K_s})$ is tamely isomorphic to an infinite type power series space of infinite type.*

Given a S^* -parabolic Stein manifold X , $\dim X = n$, with a special exhaustion function ρ , a natural grading for $O(X)$ can be obtained by considering the grading induced by the exhaustion $(\overline{D}_k)_{k=1}^\infty$ where $D_k = (z : \rho(z) < k)$, $k = 1, 2, \dots$, are the sub-level sets of ρ . We will conclude this section with a result about the Fréchet space structure of this graded space.

THEOREM 2.14. ([7]) *With the above notation the graded Fréchet space $(O(X), \|\cdot\|_{\overline{D}_s})$ is tamely isomorphic to an infinite type power series space $\Lambda_\infty(\alpha)$ where the sequence $\alpha = (\alpha_n)_n$ satisfies*

$$\lim_m \frac{\alpha_m}{m^{\frac{1}{n}}} = 2\pi(n!)^{\frac{1}{n}} \left(\int_X (dd^c \rho)^n \right)^{-\frac{1}{n}}.$$

3. Aspects of pluripotential theory on S -parabolic manifolds

The complex space \mathbb{C}^n with the special exhaustion function $\log |z|$ is a classical and inspiring example of a parabolic manifold. One can introduce a pluripotential theory on a S -parabolic manifold (X, ρ) by taking the well-studied complex pluripotential theory on \mathbb{C}^n as a model and by using ρ instead of $\log |z|$. On S^* -parabolic manifolds, analogs of basic notions of classical pluripotential theory were introduced by Zeriahi [38] (see also [7]). In this section we introduce the analog of classical Lelong classes for parabolic manifolds with not-necessarily continuous special exhaustion functions i.e. for S -parabolic manifolds and consider certain plurisubharmonic functions belonging to this class.

DEFINITION 3.1. Let (X, ρ) be a S -parabolic manifold. The class

$$\mathcal{L}_\rho = \{u(z) \in PSH(X) : u(z) \leq c_u + \rho^+(z) \forall z \in X\},$$

where c_u is a constant, $\rho^+(z) = \max\{0, \rho(z)\}$, will be called the *Lelong class* corresponding to the special exhaustion function ρ . By $\mathcal{L}_\rho(K)$, $K \subset X$ a compact set, we denote the class

$$\mathcal{L}_\rho(K) = \{u \in \mathcal{L}_\rho : u|_K \leq 0\}.$$

The analog of Zaharyuta-Siciak extremal function for this class i.e. the upper regularization $V^*(z, K) = \lim V(z, K)$ of $V(z, K) = \sup\{u(z) \in \mathcal{L}(K)\}$ will be called the ρ -Green function of K .

Note that $V^*(z, K)$ could either be identically $+\infty$ (if K is pluripolar) or it belongs to \mathcal{L}_p and defines a special exhaustion function for X (if K is not pluripolar).

Pluriregular points, for a compact $K \subset X$, can be defined, in accordance with the classical case, as the points $z_0 \in X$ for which $V^*(z, K) = 0$. A compact set $K \subset X$ will be called *pluriregular* in case all of its points are pluriregular i.e. $V^*(z, K) = 0 \forall z \in K$. It is not difficult to show, arguing as in the classical case, that the closure \overline{D} of a domain $D \subset X$ with the piecewise smooth boundary, $\partial D \in C^1$, is pluri-regular. Consequently there is a rich supply of pluri-regular compact set for a given S -parabolic manifold.

On the space \mathbb{C}^n it is a classical fact due to Zaharyuta that for a compact pluriregular set, $V(z, K)$ is a continuous function (see [14]). Zeriahi observed that the same result is valid for S^* -parabolic manifolds [38]. On the other hand for a S -parabolic manifold X if $V^*(z, K) \in C(X)$ for a compact $K \subset\subset X$, then X becomes a S^* -parabolic manifold. In fact in this case one can take $V^*(z, K)$ as a special exhaustion function for X .

Our next theorem gives a criterion for checking continuity of $V^*(z, K)$ for pluriregular compact subsets of a S -parabolic manifold X .

THEOREM 3.2. (see [6]). *Let (X, ρ) be a S^* -parabolic manifold with special exhaustion function $\rho(z) \in psh(X)$ and let $\rho_*(z) = \varliminf_{w \rightarrow z} \rho(z)$ be the measure of discontinuity of ρ at the point $z \in X$. If*

$$(3.1) \quad \varlimsup_{\rho(z) \rightarrow \infty} \frac{\rho(z)}{\rho_*(z)} = \lim_{\rho(z) \rightarrow \infty} \frac{\rho(z)}{\rho_*(z)} = 1$$

then $V^(z, K) \in C(X)$ for any pluriregular compact $K \subset X$.*

We note, that the condition (3.1) means continuity of $\rho(z)$ at infinitive points of X .

PROOF. We fix a pluriregular compact $K \subset X$ and take the Green function $V^*(z, K)$. It is clear, that there exist a constants C_1, C_2 :

$$C_1 + \rho^+(z) \leq V^*(z, K) \leq C_2 + \rho^+(z) \quad \forall z \in X.$$

It follows, that the Green function $\nu(z) = V^*(z, K)$ also satisfies the condition (3.1).

By the approximation theorem (see [12], [25]) we can approximate $V^*(z, K) \in psh(X)$: we can find a sequence of smooth psh functions

$$\nu_j(z) \in psh(X) \cap C^\infty(X), \quad \nu_j(z) \downarrow \nu(z), \quad z \in X.$$

Since $K \subset X$ is pluriregular, then $\nu|_K \equiv 0$ and for fixed $\varepsilon > 0$ we take the neighborhood $U = \{\nu(z) < \varepsilon/2\} \supset K$. Applying for $K \subset U$ the well-known Hartog's lemma to $\nu_j(z) \downarrow \nu(z)$, we have:

$$\nu_j(z) < \varepsilon, \quad \forall j \geq j_0, \quad z \in K.$$

By (3.1) there exists $R > 0$ such that

$$(3.2) \quad \nu(z) < \nu_*(z) + \varepsilon \nu_*(z), \quad z \notin B_R = \{z \in X : \nu(z) < R\}, \quad B_R \supset K.$$

If $z \in \partial B_R$, then by (3.2), $\nu(z) < (1 + \varepsilon)\nu_*(z) \leq (1 + \varepsilon)R$. Applying again the Hartog's lemma we have

$$\nu_j(z) < (1 + 2\varepsilon)R, \quad j > j_1 \geq j_0, \quad z \in \partial B_R.$$

Fix $j > j_1$ and put

$$w(z) = \begin{cases} \max\{\nu_j(z), (1 + 3\varepsilon)\nu(z)\} & \text{if } z \in B_R, \\ (1 + 3\varepsilon)\nu(z) - \varepsilon R & \text{if } z \notin B_R. \end{cases}$$

Since for $z \in \partial B_R$ we have $w(z) = (1 + 3\varepsilon)\nu(z) - \varepsilon R \geq (1 + 3\varepsilon)R - \varepsilon R = (1 + 2\varepsilon)R \geq \nu_j(z)$, then $w(z) \in psh(X)$. Hence, the function

$$\frac{1}{1 + 3\varepsilon}(w(z) - \varepsilon) \in \mathcal{L}.$$

Since for $z \in K$ this function is negative, then

$$\frac{1}{1 + 3\varepsilon}(w(z) - \varepsilon) \leq V^*(z, K).$$

It follows, that $\nu_j(z) \leq (1 + 3\varepsilon)V^*(z, K) + \varepsilon$, $z \in B_R$. This with $\nu_j(z) \geq V^*(z, K)$ gives continuity of $V^*(z, K)$ in B_R and consequently on X . \square

Note that Theorem 3.2 follows, that in the condition (3.1) X is S^* -parabolic. On the other hand if X is S^* -parabolic, i.e. ρ is continuous, then $\rho(z) \equiv \rho_*(z)$, so the condition (3.1) is satisfied automatically.

We will now introduce the main objects of our study, namely the polynomials on S -parabolic manifolds.

DEFINITION 3.3. Let (X, ρ) be a S -parabolic manifold. A holomorphic function $f \in O(X)$ is called a *polynomial* on X in case for some integers d and $c > 0$ f satisfies the growth estimate

$$\ln |f(z)| \leq d \cdot \rho^+(z) + c \quad \forall z \in X.$$

The minimal such d will be called the *degree* of f and the set of all polynomials on X with degree less than or equal to d will be denoted by \mathcal{P}_ρ^d .

A. Zeriahi, using an idea of Plesniak [19] showed that the vector spaces \mathcal{P}_ρ^d , for an S -parabolic manifold is finite dimensional, and give bounds for their dimension [38]. We will give a different proof of this result using techniques of [6].

THEOREM 3.4. *Let (X, ρ) be an S -parabolic Stein manifold. The space \mathcal{P}_ρ^d is a finite dimensional complex vector space and there exists a $C = C(X) > 0$ such that $\dim \mathcal{P}_\rho^d \leq Cd$.*

SKETCH OF THE PROOF. Let us choose $\delta(d)$ linearly independent elements from \mathcal{P}_ρ^d . Fix a pluriregular compact set K and any domain D with $K \subset D \subset\subset X$. We choose an $R_D \in \mathbb{N}$ such that $\overline{D} \subset \{z : V^*(z, K) < R_D\}$.

Any polynomial p of degree less than or equal to d , satisfies

$$\frac{1}{d} \ln \left(\frac{|p(z)|}{\|p\|_K} \right) \leq V^*(z, K) \quad \forall z \in X.$$

The norm's we will use in this proof are the sup norms.

In particular we have

$$\|p\|_{\overline{D}} \leq e^{d \cdot R_D} \|p\|_K \quad \forall p \in \mathcal{P}_\rho^d.$$

At this point we will put to use two results from functional analysis: the first is the well-known theorem of Tichomirov which in our setting says that the above estimate yields an estimate from below of the $\delta(d) - 1$ 'th Kolmogorov diameter in $C(K)$ of the restriction of the unit ball $O(X)|_{\overline{D}} \subset C(\overline{D})$ to K and a general fact from [3] that says it is possible to choose a D for this K such that the sequence of Kolmogorov diameters considered above is weakly asymptotic $\{e^{-m^{1/n}}\}$. We refer the reader to [4] for details. By choosing D suitable, one gets

$$\exists C_1 > 0 : e^{-(\delta(d)-1)^{1/n}} \geq C_1 e^{d \cdot R_D}.$$

Hence

$$\exists C_2 > 0 : \delta(d) \leq C_2 d^n \text{ for all } d = 1, 2, \dots$$

□

In the case of algebraic affine manifolds of dimension n with canonical special exhaustion function, we actually have that the sequence $\{\dim \mathcal{P}_\rho^d\}_d$ and $\{d^n\}_d$ are weakly asymptotic i.e. $\exists C_1 > 0$ and $C_2 > 0$ such that

$$C_1 \leq \underline{\lim}_{d \rightarrow \infty} \frac{\dim \mathcal{P}_\rho^d}{d^{1/n}} \leq \overline{\lim}_{d \rightarrow \infty} \frac{\dim \mathcal{P}_\rho^d}{d^{1/n}} \leq C_2.$$

For more information on these matters we refer to the reader to [40] and [6].

4. Example

In this section we will construct a parabolic manifold for which there are no non-trivial polynomials. In the first part of the section we will first construct a compact polar set $K \subset \mathbb{C}$ and a subharmonic function $u(z)$ on the complex plane \mathbb{C} , harmonic in $\mathbb{C} \setminus K$, for which $u|_K = -\infty$ and

$$\lim_{z \rightarrow K} \frac{u(z)}{\ln \text{dist}(z, K)} = 0.$$

The condition above means, in particular, that near K , the function $|u(z)|$ is smaller than $\varepsilon |\ln \text{dist}(z, K)|$. We note that for compact sets containing an isolated point, such that function does not exists.

In the second part of the section we will use this example to construct

THEOREM 4.1. *There exists a polar compact $K \subset \mathbb{C}$ and a subharmonic function $u(z)$ on the complex plane \mathbb{C} , harmonic in $\mathbb{C} \setminus K$, for which $u|_K = -\infty$, and*

$$(4.1) \quad \lim_{z \rightarrow K} \frac{u(z)}{\ln \text{dist}(z, K)} = 0.$$

PROOF. We take a special Cantor set $K \in [0, 1] \subset \mathbb{C}^n$ and the probability measure μ , $\text{supp} \mu \subset K$ on it such that, the potential of μ tends to $-\infty$ slowly than any $\varepsilon \ln \text{dist}(z, K) \forall \varepsilon > 0$.

Consider the segment $[0, 1]$, and denote it as $K_0 = [a_{01}, b_{01}]$, the length of K_0 is 1. Next we proceed as in the construction of Cantor sets: fix $\delta = 1/4$ and the sequence $t_m = 4^{m-1}$, $m = 1, 2, \dots$. From (a_{01}, b_{01}) we put off the interval $[a_{01} + \delta, b_{01} - \delta]$. We get the union of two segments, $K_1 = [a_{01}, a_{01} + \delta] \cup [a_{01} - \delta, a_{01}]$. Redenote them as $K_1 = [a_{01}, a_{01} + \delta] \cup [b_{01} - \delta, b_{01}] = [a_{11}, b_{11}] \cup [a_{12}, b_{12}]$. Distances between knot-points $a_{11}, b_{11}, a_{12}, b_{12}$ are:

$$|b_{1j} - a_{1j}| = \delta, j = 1, 2, \quad |b_{11} - a_{12}| = 1 - 2\delta.$$

Then with each of these segments we do the same procedure, changing δ to δ^{t_2} : we get 4 segments,

$$\begin{aligned} K_2 &= [a_{11}, a_{11} + \delta^{t_2}] \cup [b_{11} - \delta^{t_2}, b_{11}] \cup [a_{12}, a_{12} + \delta^{t_2}] \cup [b_{12} - \delta^{t_2}, b_{12}] = \\ &= [a_{21}, b_{21}] \cup [a_{22}, b_{22}] \cup [a_{23}, b_{23}] \cup [b_{24}, b_{24}], \end{aligned}$$

with length δ^{t_2} , and with distances between knot points:

$$\begin{aligned} |b_{2j} - a_{2j}| &= \delta^{t_2}, j = 1, 2, 3, 4, \\ |b_{21} - a_{22}| &= \delta - 2\delta^{t_2}, |b_{22} - a_{23}| = 1 - 2\delta, |b_{23} - a_{24}| = \delta - 2\delta^{t_2}. \end{aligned}$$

In m -th step we get union of 2^m segments

$$K_m = [a_{m1}, b_{m1}] \cup [a_{m2}, b_{m2}] \cup \dots \cup [a_{m2^m}, b_{m2^m}],$$

with length δ^{t_m} . Note, $K_0 \supset K_1 \supset \dots \supset K_m$, $l(K_m) = 2^m \delta^{t_m}$. Moreover, the Hausdorff measure of K_m with respect to kernel $h(s) = \ln^{-1} \frac{1}{s}$ is equal to

$$(4.2) \quad H^h(K_m) = 2^m h(\delta^{t_m}/2) = 2^m \ln^{-1} \frac{1}{\delta^{t_m}/2} = \frac{2^m}{t_m} \ln^{-1} \frac{2^{1/t_m}}{\delta}.$$

Put $K = \bigcap_{m=1}^{\infty} K_m$. If $\frac{2^m}{t_m} \leq C < \infty$, $m = 1, 2, \dots$, then $H^h(K) < \infty$ and by the well-known property of the logarithm capacity $C(K) = 0$. Therefore, in our case $t_m = 4^{m-1}$, the compact set K is polar and there exists a probability measure μ , $\text{supp} \mu = K$, such that its potential

$$U^\mu(z) = \int \ln |z - w| d\mu(w)$$

is harmonic off K , subharmonic on \mathbb{C}^n , and $U^\mu(z) = -\infty \forall z \in K$.

Now we will specifically construct such measure μ . For $K_m = [a_{m1}, b_{m1}] \cup [a_{m2}, b_{m2}] \cup \dots \cup [a_{m2^m}, b_{m2^m}]$ we put

$$(4.3) \quad \mu_m = \frac{\delta(a_{m1}) + \dots + \delta(a_{m2^m}) + \delta(b_{m1}) + \dots + \delta(b_{m2^m})}{2 \cdot 2^m},$$

where $\delta(c)$ -discrete probability measure, supported in c . The sequence μ_m weakly tends to a measure $\mu_m \mapsto \mu$, $\text{supp} \mu = K$. Let

$$U_m^\mu(z) = \int \ln |z - w| d\mu_m(w), U^\mu(z) = \int \ln |z - w| d\mu(w)$$

be the potentials. We give some estimations to these potentials.

Take $z^0 \in \mathbb{C}^n \setminus K$, $\lambda = \text{dist}(z^0, K) > 0$. Then by a well-known integral formula (see [11]).

$$(4.4) \quad U_m^\mu(z^0) = \int \ln |z^0 - w| d\mu_m(w) = \int_0^\infty [\ln t] d\mu_m(z^0, t) = \int_{\lambda_m}^\Lambda [\ln t] d\mu_m(z^0, t),$$

where $\mu_m(z^0, t) = \mu_m(B(z^0, t))$, $B(z^0, t) : |z - z^0| \leq t$ is disk, $\Lambda = \max\{\text{dist}(z^0, 0), \text{dist}(z^0, 1)\}$, $\lambda_m = \min\{|z^0 - a_{mj}|, |z^0 - b_{mj}| : j = 1, 2, \dots, 2^m\}$ is the distance from z^0 to the knot set $K_m^{\text{knot}} = \{a_{m1}, b_{m1}, a_{m2}, b_{m2}, \dots, a_{m2^m}, b_{m2^m}\}$, $\lambda_m \geq \lambda$. Integrating by part (4.4) we get

$$\begin{aligned} U_m^\mu(z^0) &= \int_{\lambda_m}^\Lambda [\ln t] d\mu_m(z^0, t) = \mu_m(z^0, t) \ln t|_{\lambda_m}^\Lambda - \int_{\lambda_m}^\Lambda \frac{\mu_m(z^0, t)}{t} dt = \\ &= \ln \Lambda - \int_{\lambda_m}^\Lambda \frac{\mu_m(z^0, t)}{t} dt. \end{aligned}$$

Next we will estimate the potentials $U_m^\mu(z^0), U^\mu(z^0)$ for nearby to K point z^0 , say $\lambda_m < 1$. Let c is a knot point, such that $\lambda_m = |z^0 - c|$. The cases $c = 0$ or $c = 1$ are simple and both are similar one to one. Other cases reduces to these cases by parting knot set $\{a_{m1}, b_{m1}, a_{m2}, b_{m2}, \dots, a_{m2^m}, b_{m2^m}\}$ two sets: right and left from $\text{Re } z^0$. Therefore, without loss of generality, we assume that $c = 0$ and $\text{Re } z^0 \leq 0$. In this case, $\mu_m(0, t - \lambda_m) \leq \mu_m(z^0, t) \leq \mu_m(0, \sqrt{t^2 - \lambda_m^2})$. If we denote $\mu_m(t) = \mu_m(0, t)$, then

$$(4.5) \quad - \int_{\lambda_m}^{\Lambda} \frac{\mu_m(t - \lambda_m)}{t} dt \leq - \int_{\lambda_m}^{\Lambda} \frac{\mu_m(z^0, t)}{t} dt \leq - \int_{\lambda_m}^{\Lambda} \frac{\mu_m(\sqrt{t^2 - \lambda_m^2})}{t} dt$$

It is clear, that

$$\mu_m(\delta) = \frac{1}{2}, \mu_m(\delta^{t_2}) = \frac{1}{2^2}, \dots, \mu_m(\delta^{t_{m-1}}) = \frac{1}{2^{m-1}}, \mu_m(\delta^{t_m}) = \frac{1}{2^m}.$$

Therefore,

$$(4.6) \quad \begin{aligned} \mu_m(t) &= \frac{1}{2}, \text{ if } \delta \leq t < 1 - \delta; \\ \mu_m(t) &= \frac{1}{2^2}, \text{ if } \delta^{t_2} \leq t < \delta - \delta^{t_2}; \\ &\vdots \\ \mu_m(t) &= \frac{1}{2^{m-1}}, \text{ if } \delta^{t_{m-1}} \leq t < \delta^{t_{m-2}} - \delta^{t_{m-1}}; \\ \mu_m(t) &= \frac{1}{2^m}, \text{ if } \delta^{t_m} \leq t < \delta^{t_{m-1}} - \delta^{t_m}. \end{aligned}$$

Using (4.5) and (4.6) we can give upper and lower bounds of $U^\mu(z)$.

a) Upper bound of the potential $U^\mu(z)$. We have

$$(4.7) \quad \begin{aligned} I_m &= - \int_{\lambda_m}^{\Lambda} \frac{\mu_m(z^0, t)}{t} dt \leq - \int_{\lambda_m}^{\Lambda} \frac{\mu_m(\sqrt{t^2 - \lambda_m^2})}{t} dt = - \int_0^{\sqrt{\Lambda^2 - \lambda_m^2}} \frac{t}{t^2 + \lambda_m^2} \mu_m(t) dt - \\ &- \int_0^{\delta^{t_{m-1}} - \delta^{t_m}} \frac{t}{t^2 + \lambda_m^2} \mu_m(t) dt - \int_{\delta^{t_{m-1}} - \delta^{t_m}}^{\delta^{t_{m-2}} - \delta^{t_{m-1}}} \frac{t}{t^2 + \lambda_m^2} \mu_m(t) dt - \dots - \int_{\delta - \delta^{t_2}}^{1 - \delta} \frac{t}{t^2 + \lambda_m^2} \mu_m(t) dt - \\ &- \int_{1 - \delta}^1 \frac{t}{t^2 + \lambda_m^2} \mu_m(t) dt - \int_{\delta^{t_m}}^{\delta^{t_{m-1}} - \delta^{t_m}} \frac{t}{t^2 + \lambda_m^2} \mu_m(t) dt - \int_{\delta^{t_{m-1}}}^{\delta^{t_{m-2}} - \delta^{t_{m-1}}} \frac{t}{t^2 + \lambda_m^2} \mu_m(t) dt - \dots \\ &\dots - \int_{\delta}^{1 - \delta} \frac{t}{t^2 + \lambda_m^2} \mu_m(t) dt = - \frac{2}{2^{m+1}} \int_{\delta^{t_m}}^{\delta^{t_{m-1}} - \delta^{t_m}} \frac{t dt}{t^2 + \lambda_m^2} - \frac{2^2}{2^{m+1}} \int_{\delta^{t_{m-1}}}^{\delta^{t_{m-2}} - \delta^{t_{m-1}}} \frac{t dt}{t^2 + \lambda_m^2} - \dots \\ &\dots - \frac{2^m}{2^{m+1}} \int_{\delta}^{1 - \delta} \frac{t dt}{t^2 + \lambda_m^2}. \end{aligned}$$

Therefore

$$\begin{aligned}
I_m &\leq \frac{2}{2^{m+2}} \ln \frac{\lambda_m^2 + \delta^{2t_m}}{\lambda_m^2 + (\delta^{t_{m-1}} - \delta^{t_m})^2} + \frac{2^2}{2^{m+2}} \ln \frac{\lambda_m^2 + \delta^{2t_{m-1}}}{\lambda_m^2 + (\delta^{t_{m-2}} - \delta^{t_{m-1}})^2} + \dots \\
&\dots + \frac{2^m}{2^{m+1}} \ln \frac{\lambda_m^2 + \delta^2}{\lambda_m^2 + (1 - \delta)^2} = \frac{1}{2^m} \ln \frac{\lambda_m^2 + \delta^{2t_m}}{\lambda_m^2 + \delta^{2t_{m-1}}} + \frac{2}{2^m} \ln \frac{\lambda_m^2 + \delta^{2t_{m-1}}}{\lambda_m^2 + \delta^{2t_{m-2}}} + \dots \\
(4.8) \quad &\dots + \frac{2^{m-1}}{2^m} \ln \frac{\lambda_m^2 + \delta^2}{\lambda_m^2 + 1} + o(\delta^{t_{m-1}}) = \frac{1}{2^m} \ln(\lambda_m^2 + \delta^{2t_m}) \frac{1}{2^m} \ln(\lambda_m^2 + \delta^{2t_{m-1}}) + \\
&+ \frac{2}{2^m} \ln(\lambda_m^2 + \delta^{2t_{m-2}}) + \dots \frac{2^{m-2}}{2^m} \ln(\lambda_m^2 + \delta^2) - \frac{2^{m-1}}{2^m} \ln(\lambda_m^2 + 1) + o(\delta^{t_{m-1}}).
\end{aligned}$$

Let $k = k(z^0)$ be the smallest natural number, such that $\delta^{t_k} \leq \lambda_m$. We part the last sum in (4.8) into two sums: by $k \leq j \leq m$ ($\delta^{t_j} \leq \lambda_m$) and by $j < k$ ($\delta^{t_j} > \lambda_m$). For the first sum, by $\delta^{t_j} \leq \lambda_m$, we write

$$\begin{aligned}
&\frac{1}{2^m} \ln(\lambda_m^2 + \delta^{2t_m}) + \frac{1}{2^m} \ln(\lambda_m^2 + \delta^{2t_{m-1}}) + \frac{2}{2^m} \ln(\lambda_m^2 + \delta^{2t_{m-2}}) + \dots \\
&\dots + \frac{2^{m-k-1}}{2^m} \ln(\lambda_m^2 + \delta^{2t_m}) \leq \frac{1 + 2 + \dots + 2^{m-k-1}}{2^m} \ln(2\lambda_m^2) = \frac{2^{m-k} - 1}{2^m} \ln(2\lambda_m^2) \leq \frac{1}{2^m} \ln(2\lambda_m^2).
\end{aligned}$$

Since $t_k = 4^{k-1}$ and $\delta^{t_k} \leq \lambda_m$, then $2^k \geq \sqrt{\frac{\ln \lambda_m}{\ln \delta}}$. Therefore, the first sum is not greater than $\frac{1}{2^k} \ln 2\lambda_m^2 \leq \sqrt{\ln \frac{1}{\delta}} \frac{2 \ln \lambda_m + \ln 2}{\sqrt{\ln \frac{1}{\lambda_m}}}$.

For the second sum, by $\delta^{t_j} > \lambda_m$, we have,

$$\begin{aligned}
&\frac{2^{m-k}}{2^m} \ln(\lambda_m^2 + \delta^{2t_{k-1}}) + \dots + \frac{2^{m-2}}{2^m} \ln(\lambda_m^2 + \delta^2) - \frac{2^{m-1}}{2^m} \ln(\lambda_m^2 + 1) \leq -\frac{1}{2} \ln(\lambda_m^2 + 1) + \\
&+ \frac{1}{2^2} \ln(\lambda_m^2 + \delta^2) + \dots + \frac{1}{2^k} \ln(\lambda_m^2 + \delta^{2t_{k-1}}) \leq -\frac{1}{2} \ln(\lambda_m^2 + 1) + \frac{1}{2^2} \ln(2\delta^2) + \dots + \frac{1}{2^k} \ln(2\delta^{2t_{k-1}}) \leq \\
&\leq -\frac{1}{2} \ln(\lambda_m^2 + 1) + \frac{1}{2^2} \ln(2\delta^2) + \dots + \frac{1}{2^{k+1}} \ln(2\delta^2) = -\frac{1}{2} \ln(\lambda_m^2 + 1) + \frac{1}{2} \ln(2\delta^2).
\end{aligned}$$

Therefore, for large enough m is true the following estimation

$$(4.9) \quad U_m^\mu(z^0) \leq \sqrt{\ln \frac{1}{\delta}} \frac{\ln \lambda_m + \ln 2}{\sqrt{\ln \frac{1}{\lambda_m}}} - \frac{1}{2} \ln(\lambda_m^2 + 1) + \ln \Lambda + \frac{1}{2} \ln 2\delta + o(\delta^{t_{m-1}}).$$

For arbitrary $z^0 \in \mathbb{C}^n \setminus K$ the estimation (4.9) will be

$$\begin{aligned}
(4.10) \quad U_m^\mu(z^0) &\leq 2\sqrt{\ln \frac{1}{\delta}} \frac{\ln \text{dist}(z^0, K_m^{\text{knot}}) + \ln 2}{\sqrt{\ln \frac{1}{\text{dist}(z^0, K_m^{\text{knot}})}} - \\
&- \frac{1}{2} \ln(\text{dist}^2(z^0, K_m^{\text{knot}}) + 1) + \ln \Lambda + \frac{1}{2} \ln 2\delta + o(\delta^{t_{m-1}}).
\end{aligned}$$

Tending $m \rightarrow \infty$ in (4.10) we take

$$U^\mu(z^0) \leq 2\sqrt{\ln \frac{1}{\delta}} \frac{\ln \text{dist}(z^0, K) + \ln 2}{\sqrt{\ln \frac{1}{\text{dist}(z^0, K)}} -$$

$$(4.11) \quad -\frac{1}{2} \ln(\text{dist}^2(z^0, K) + 1) + \ln \Lambda + \frac{1}{2} \ln 2\delta.$$

From (4.11), in particular, follows, that $U^\mu(z^0) = -\infty, \forall z^0 \in K$.

b) Lower bound. As above, we have:

$$\begin{aligned} I_m &= - \int_{\lambda_m}^{\Lambda} \frac{\mu_m(z^0, t)}{t} dt \geq - \int_{\lambda_m}^{\Lambda} \frac{\mu_m(t - \lambda_m)}{t} dt = - \int_0^{\Lambda - \lambda_m} \frac{\mu_m(t)}{t + \lambda_m} dt = \\ &= - \int_0^{\delta^{t_{m-1}} - \delta^{t_m}} \frac{\mu_m(t)}{t + \lambda_m} dt - \int_{\delta^{t_{m-1}} - \delta^{t_m}}^{\delta^{t_{m-2}} - \delta^{t_{m-1}}} \frac{\mu_m(t)}{t + \lambda_m} dt - \dots - \int_{\delta - \delta^{t_2}}^{1 - \delta} \frac{\mu_m(t)}{t + \lambda_m} dt - \int_{1 - \delta}^1 \frac{\mu_m(t)}{t + \lambda_m} dt \geq \\ &\geq - \frac{2}{2^{m+1}} \int_0^{\delta^{t_{m-1}} - \delta^{t_m}} \frac{dt}{t + \lambda_m} - \frac{2^2}{2^{m+1}} \int_{\delta^{t_{m-1}} - \delta^{t_m}}^{\delta^{t_{m-2}} - \delta^{t_{m-1}}} \frac{dt}{t + \lambda_m} - \dots - \frac{2^m}{2^{m+1}} \int_{\delta - \delta^{t_2}}^{1 - \delta} \frac{dt}{t + \lambda_m} - \frac{2^{m+1}}{2^{m+1}} \int_{1 - \delta}^1 \frac{dt}{t + \lambda_m} = \\ &= - \frac{1}{2^m} \ln \frac{\lambda_m + \delta^{t_{m-1}} - \delta^{t_m}}{\lambda_m} - \frac{1}{2^{m-1}} \ln \frac{\lambda_m + \delta^{t_{m-2}} - \delta^{t_{m-1}}}{\lambda_m + \delta^{t_{m-1}} - \delta^{t_m}} - \dots - \frac{1}{2} \ln \frac{\lambda_m + 1 - \delta}{\lambda_m + \delta - \delta^{t_2}} - \ln \frac{\lambda_m + 1}{\lambda_m + 1 - \delta} = \\ &= \frac{\ln \lambda_m}{2^m} + \frac{\ln(\lambda_m + \delta^{t_{m-1}} - \delta^{t_m})}{2^m} + \frac{\ln(\lambda_m + \delta^{t_{m-2}} - \delta^{t_{m-1}})}{2^{m-1}} + \dots + \frac{\ln(\lambda_m + 1 - \delta)}{2} - \ln(\lambda_m + 1). \end{aligned}$$

Therefore

$$\begin{aligned} I_m &\geq -\ln(\lambda_m + 1) + \frac{\ln(\lambda_m + 1 - \delta)}{2} + \frac{\ln(\lambda_m + \delta - \delta^{t_2})}{2^2} + \\ &+ \frac{\ln(\lambda_m + \delta^{t_2} - \delta^{t_3})}{2^3} + \dots + \frac{\ln(\lambda_m + \delta^{t_{m-2}} - \delta^{t_{m-1}})}{2^{m-1}} + \frac{\ln(\lambda_m + \delta^{t_{m-1}} - \delta^{t_m})}{2^m} + \frac{\ln \lambda_m}{2^m} \geq \\ &\geq -\ln(\lambda_m + 1) + \frac{\ln(1 - \delta)}{2} + \frac{\ln(\delta - \delta^{t_2})}{2^2} + \dots + \frac{\ln(\delta^{t_{k-1}} - \delta^{t_k})}{2^k} + \\ &+ \frac{\ln \lambda_m}{2^{k+1}} + \dots + \frac{\ln \lambda_m}{2^{m-1}} + \frac{\ln \lambda_m}{2^m} + \frac{\ln \lambda_m}{2^m} = c(k) + \frac{\ln \lambda_m}{2^k} \left(1 - \frac{1}{2^{m-k}}\right), \end{aligned}$$

where $c(k) = \text{const}$, independent of m . Hence, for any fixed $k \in \mathbb{C}^n$ we have

$$(4.12) \quad U_m^\mu(z^0) \geq \ln \Lambda + c(k) + \frac{\ln \lambda_m}{2^k} \left(1 - \frac{1}{2^{m-k}}\right).$$

As above we can prove (4.12) for arbitrary $z^0 \notin K$:

$$(4.13) \quad U_m^\mu(z^0) \geq 2 \left(\ln \Lambda + c(k) + \frac{\ln \text{dist}(z^0, K_m^{\text{knot}})}{2^k} \left(1 - \frac{1}{2^{m-k}}\right) \right).$$

Tending $m \rightarrow \infty$ from (4.13) we conclude, that for any $\varepsilon > 0$ there exists constant $c(\varepsilon) > -\infty$:

$$U_m^\mu(z^0) \geq c(\varepsilon) + \varepsilon \ln \text{dist}(z^0, K), \forall z^0 \in \mathbb{C}^n.$$

Theorem is proved. \square

Now we can proceed with our example,

EXAMPLE 4.2. We consider the manifold $X = \overline{\mathbb{C}} \setminus K$, where K is compact, built in the previous point. As special exhaustive function we put $\phi(z) = -U^\mu(z)$. Then $\phi(z)$ is harmonic on $X \setminus \{\infty\}$, $\phi(\infty) = -\infty$ and $\phi(z) \rightarrow \infty$ as $z \rightarrow K$. Therefore, (X, ϕ) is S^* -parabolic.

Polynomials on X are functions $f \in O(X)$ for which $\ln |f| \leq C + d\phi(z)$, $d \in \mathbb{N}$. We show that this like functions are trivial, i.e. $f = \text{const}$. It follows, that on X there are not nontrivial polynomials, X is nonregular.

This easily follows from the next Proposition, which seems clear and there is a proof of them: let K is a polar compact on the complex plane \mathbb{C} , where $U \supset K$ is some neighborhood. If $f(z) \in O(U \setminus K)$ and

$$(4.14) \quad \overline{\lim}_{z \rightarrow K} |f(z)| \cdot \text{dist}(z, K) = 0,$$

then $f(z) \in O(U)$.

Since we cannot find the proof of this proposition, we provide it for our compact K . Let $f \in O(X) : \ln |f| \leq C + k\phi(z)$.

First we take a closed curve $\gamma = \gamma_m$, containing within itself the $K \subset K_m = [a_{m1}, b_{m1}] \cup [a_{m2}, b_{m2}] \cup \dots \cup [a_{22m}, b_{22m}] : \gamma$ bounds above by a part of $\{\text{Im} z = r\}$, $r > 0$, below by $\{\text{Im} z = -r\}$ and from the sides by a part $\{\text{Re} z = a_{mj} - r\}$, $\{\text{Re} z = b_{mj} + r\}$. The length of γ is equal

$$(4.15) \quad l(\gamma) = 2 \cdot 2^m (\delta^{t_m} + 2r) + 2 \cdot 2^m r = 3 \cdot 2^{m+1} r + 2^{m+1} \delta^{t_m}.$$

To complete of the proof we write the Cauchy formula

$$(4.16) \quad f(z) = \frac{1}{2\pi i} \int_{|\xi|=2} \frac{f(\xi)}{\xi - z} d\xi - \frac{1}{2\pi i} \int_{\gamma} \frac{f(\xi)}{\xi - z} d\xi, \quad z \in B(0, 2) \setminus \hat{\gamma},$$

where $\hat{\gamma}$ is the polynomial convex hull of γ .

For second integral of (4.16) we have

$$\left| \int_{\gamma} \frac{f(\xi)}{\xi - z} d\xi \right| \leq \frac{\|f\|_{\gamma}}{\text{dist}(z, \gamma)} l(\gamma) \leq \frac{e^{C+k\|\phi\|_{\gamma}}}{\text{dist}(z, \gamma)} (3 \cdot 2^{m+1} r + 2^{m+1} \delta^{t_m}) \leq C_1 e^{k\|\phi\|_{\gamma}} (2^{m+3} r + 2^{m+1} \delta^{t_m}).$$

According to (4.1) for arbitrary fixed $\varepsilon > 0$ there exists $\gamma = \gamma_m$ such, that

$$\|\phi\|_{\gamma} < -\varepsilon \ln \text{dist}(\gamma, K). \text{ Therefore, } \left| \int_{\gamma} \frac{f(\xi)}{\xi - z} d\xi \right| \leq C_2 r^{-\varepsilon k} 2^m (r + \delta^{t_m}).$$

Now we choose $\varepsilon = 1/2k$ and $r = 1/2^{4m}$. Then $r^{-\varepsilon k} 2^m (r + \delta^{t_m}) = \frac{1}{2^m} + 2^{3m} \delta^{t_m} \rightarrow 0$ as $m \rightarrow \infty$. We see that, the second integral in (4.16) tends zero, which means the function

$$f(z) = \frac{1}{2\pi i} \int_{|\xi|=R} \frac{f(\xi)}{\xi - z} d\xi$$

and holomorphic in $|z| < R$. Consequently $f \in O(\overline{\mathbb{C}})$, i.e. $f \equiv \text{const}$.

5. Regular parabolic manifolds.

As we have seen in section 4 not every parabolic manifold has a large supply of polynomials. On the other hand most important examples of parabolic manifolds like affine algebraic submanifolds (with their canonical special exhaustion function), complements of zero sets of Weierstrass polynomials (see [7]) do have a rich class of

polynomials, namely in these examples polynomials are dense in the corresponding spaces of analytic functions.

EXAMPLE 5.1. Algebraic set $X \subset \mathbb{C}^N$, $\dim A = n$. In this case by the well-known theorem of W. Rudin [22], we can assume, that (after an appropriate transformation)

$$X \subset \{w = (w', w'') = (w_1, \dots, w_n, w_{n+1}, \dots, w_N) : \|w''\| < A(1 + \|w'\|^B)\},$$

where A, B are constants. Then the restriction $\rho|_X$ of the function $\rho(w) = \ln \|w'\|$ may be special exhaustion function on X . It is clear, that polynomials on X are restrictions to X of polynomials $p(w', w'')$. Therefore, $\mathcal{P}_\rho(X)$ is dense in $O(X)$.

EXAMPLE 5.2. Complement of Weierstrass algebroid set (see Theorem 2.7). Let

$$A = \{z = (z', z_n) = (z_1, z_2, \dots, z_n) \in \mathbb{C}^n : F(z', z_n) = z_n^k + f_1(z')z_n^{k-1} + \dots + f_k(z') = 0\}$$

be a Weierstrass polynomial set, where $f_j \in O(\mathbb{C}^{n-1})$ are entire functions, $j = 1, 2, \dots, k$, $k > 1$. Then $X = \mathbb{C}^n \setminus A$ with exhaustion function $\rho(z) = -\ln |F(z)| + \ln(|z'| + |F(z) - 1|^2)$ is S^* -parabolic. If $p(z, \tau)$ is a polynomial in \mathbb{C}^{n+1} , then $p(z, 1/F(z))$ is a polynomial on $X = \mathbb{C}^n \setminus A$.

It is not difficult to prove, that $\{p(z, 1/F(z))\}_p$ is dense in $O(X)$.

Motivated by these examples, we give the following definition:

DEFINITION 5.3. S^* -parabolic manifold (X, ρ) calls *regular* in case if the space of all polynomials $\mathcal{P}_\rho(X)$ is dense in $O(X)$.

Our next example shows that non triviality of the polynomial space $\mathcal{P}_\rho(X)$ does not always guarantee the regularity of X .

EXAMPLE 5.4. We add to compact K , from example 4.2 one more point: $E = K \cup \{z^0\}$, $z^0 \notin K$. The manifold $X = \overline{\mathbb{C}} \setminus E$ with exhaustive function $\rho(z) = -U^\mu(z) - \ln |z - z^0|$ be S^* -parabolic. On X there are polynomials, an example, $f(z) = (z - z^0)^m$, but the space of all polynomials \mathcal{P}_ρ is not dense in $O(X)$: the function $f(z) = \frac{1}{z - z'}$, where $z' \in K$, cannot be approximated by polynomials.

In search for more examples of S -parabolic manifolds one may look at closed complex submanifolds of regular S^* -parabolic manifolds. Since such manifolds are in particular parabolic, there exists, in view of Theorem 2.10, continuous linear extension operators for analytic functions on this submanifold to the ambient space. However the mere existence of continuous extension operators will not, in general give regularity as the example, in the previous section shows.

Recall that for a S^* -parabolic manifold (X, ρ) we will always consider, unless otherwise stated, the canonical grading on $O(X)$ given by ρ , and for a closed complex submanifold V of X we will provide $O(V)$ with the induced grading, i.e. the grading coming from the sup norms on $V \cap \{z : \rho(z) \leq k\}$, $k = 1, 2, \dots$. With this convention we have:

PROPOSITION 5.5. *Let (X, ρ) be a regular S^* -parabolic Stein manifold and let V be a closed complex submanifold of X . If there exists a tame linear extension operator from $O(V)$ into $O(X)$ then V becomes a regular S^* -parabolic manifold.*

PROOF. Fix a continuous linear extension operator $E : O(V) \rightarrow O(X)$ with the property:

$$\exists A > 0 \text{ such, that } \forall k \exists C_k > 0 : \|E(f)\|_k \leq C_k \|f\|_{k+A} \quad \forall f \in O(V).$$

Let as usual

$$\mathcal{A} = \{u(z) \in psh(V) : u(z) \leq L_u + \rho^+(z) \quad \forall z \in V, \quad u \leq 0 \text{ on } V \cap D_{A+2}\},$$

where $D_k = \{z \in X : \rho(z) < k\}$, $k = 1, 2, \dots$.

Fix a $u \in \mathcal{A}$. In view of Lelong Bremermann Lemma [9], u has a representation of the form:

$$u(z) = \overline{\lim}_{\xi \rightarrow z} \overline{\lim}_{m \rightarrow \infty} \frac{\ln |f_m(\xi)|}{\alpha_m}$$

for some $f_m \in O(V)$ and $\alpha_m \in \mathbb{N}$, $m = 1, 2, \dots$.

In view of Hartog's lemma, for each $k = 1, 2, \dots$, we can find a constant $C = C(k)$, such that

$$\|f_m\|_k \leq C e^{(k+L+1)\alpha_m}, m = 1, 2, \dots, \quad L = L_u.$$

Hence

$$\|E(f_m)\|_k \leq C e^{(k+A+L+1)\alpha_m}, \quad m = 1, 2, \dots,$$

and so the sequence of plurisubharmonic functions

$$\left\{ \frac{\ln |E(f_m(\xi))|}{\alpha_m} \right\}_m$$

is a locally bounded from above family. Let

$$\tilde{u}(z) = \overline{\lim}_{\xi \rightarrow z} \overline{\lim}_{m \rightarrow \infty} \frac{\ln |E(f_m)(\xi)|}{\alpha_m}.$$

The function \tilde{u} defines a plurisubharmonic function on X and has the growth estimate:

$$\tilde{u}(z) \leq \rho(z) + A + L + 2,$$

in view of the maximality of ρ . Since, the Green function $V^*(z, \overline{D}_1)$ on X is equal $[\rho - 1]^+$, then

$$\tilde{u}(z) \leq V^*(z, \overline{D}_1) + C_0, \quad z \in X.$$

By construction on V we have $u \leq \tilde{u}|_V$. It follows that

$$u(z) \leq V^*(z, \overline{D}_1) + C_0 \quad \forall z \in V \text{ and } u \in \mathcal{A}.$$

In particular the family \mathcal{A} is a locally bounded from above of plurisubharmonic functions on V . In view of the above considerations the free envelope

$$\tau(z) = \overline{\lim}_{\xi \rightarrow z} \sup_{u \in \mathcal{A}} u(\xi)$$

defines a plurisubharmonic function on V that is maximal outside a compact set $V \cap \overline{D}_{A+2}$ and satisfies the estimates:

$$\exists C > 0 : \rho(z) \leq \tau(z) \leq \rho(z) + C,$$

since $[\rho - (A + 2)]_V \in \mathcal{A}$. Hence τ provides a special exhaustion function for V . Moreover since the restriction of a ρ -polynomial to V is a τ -polynomial, the regularity of V follows. \square

REMARK 5.6. The existence of a tame linear extension operator as above is of course related to the tame splitting of tame short exact sequence:

$$0 \rightarrow I \rightarrow O(X) \xrightarrow{R} O(V) \rightarrow 0,$$

where R is the restriction operator and I is the ideal sheaf of V with the subspace grading induced from $O(X)$. Tame splitting of short exact sequences in the category of graded Fréchet spaces were studied by various authors. We refer the reader to [21] for a survey and for structural conditions on the underlying Fréchet nuclear spaces which ensure that short exact sequences in this category split.

REMARK 5.7. It was shown in [5] that in \mathbb{C}^N closed complex submanifolds that admit tame extension operators are precisely the affine algebraic submanifolds of \mathbb{C}^N . Since there are non algebraic regular S^* -parabolic Stein manifolds of \mathbb{C}^N , the statement of the Proposition is not an if and only if statement.

Our next result deals with the linear topological structure of the graded space of analytic functions $(O(X), \rho)$ on a S^* -parabolic Stein manifold (X, ρ) . Recall that for a given S^* -parabolic Stein manifold (X, ρ) , we will always assume that the special exhaustion function ρ is maximal outside a compact set that lies in $\{z : \rho(z) < 0\}$ and equip the Fréchet space $O(X)$ with the grading $(\|\cdot\|_k)_{k=1}^\infty$:

$$\|f\|_k = \sup_{z \in D_k} |f(z)|,$$

where $D_k = \{z : \rho(z) < k\}$, $k=1,2,\dots$. On $O(\mathbb{C}^n)$ the canonical grading will be the one coming from the norm system

$$\|f\|_k = \sup_{\|z\| \leq e^k} |f(z)|, \quad k = 1, 2, \dots$$

We have seen that with a suitable special exhaustion function ρ , $(O(X), \rho)$ is tamely isomorphic to $O(\mathbb{C}^n)$ with the canonical grading. Unfortunately tame isomorphisms between S^* -parabolic Stein manifolds do not necessarily map polynomials into polynomials even when the spaces are regular as the multiplication operator with the exponential function on $O(\mathbb{C})$ shows. However our next result states that for a regular S^* -parabolic Stein manifold (X, ρ) there exists a positive constant C and a tame isomorphism T from $O(\mathbb{C}^n)$, $n = \dim X$, onto $(O(X), C\rho)$ that maps polynomials into ρ -polynomials.

In the proof below we will repeatedly use a fact from functional analysis, namely the *Dynin-Mitiagin* theorem which states that if a nuclear Fréchet space $(Y, \|\cdot\|_k)$ has a basis $\{g_m\}$, then it is isomorphic, via the correspondence

$$\sum x_m g_m \leftrightarrow (x_m)_m,$$

to the Köthe space:

$$(K, |\cdot|_k) = \left\{ x = (x_m)_m : |x|_k = \sum |x_m| \|g_m\|_k < \infty, \forall k = 1, 2, \dots \right\}.$$

As usual, for sequences of real numbers $\{\alpha_k\}$ and $\{\beta_k\}$ the notation $\alpha_k \prec \beta_k$ means that there exists a constant $c > 0$ that does not depend upon k , such that $\alpha_k \prec c\beta_k$, $\forall k$.

THEOREM 5.8. *Let (X, ρ) be a regular S^* -parabolic Stein manifold. There exists a polynomial basis $\{p_m\}$ for $O(X)$ and a $C > 0$, such that the linear transformation T defined through $T(p_m) = z^{\sigma(m)}$, $m = 1, 2, \dots$, gives a tame isomorphism between $(O(X), C\rho)$ and $O(\mathbb{C}^n)$ with the usual grading.*

PROOF. We choose a Hilbert space H_0 with

$$O(\{z : \rho \leq 0\}) \hookrightarrow H_0 \hookrightarrow O(\{z : \rho < 0\}) \cap C(\{z : \rho \leq 0\}).$$

In view of Corollary 1 of [7], and the construction of the proof of Th.1.5 [36] on which the proof of the corollary depends, we can without loss of generality assume that there is a tame isomorphism $S : O(\mathbb{C}^n) \rightarrow (O(X), \rho)$ such that the sequence $\{f_m = S(z^{\sigma(m)})\}$ forms an orthonormal basis for H_0 .

Now we will choose and fix a bijection σ , between \mathbb{N} and \mathbb{N}^n satisfying $|\sigma(n)| \leq |\sigma(n+1)|$, $\forall n \in \mathbb{N}$. Observe that the identity operator gives a tame isomorphism between $O(\mathbb{C}^n)$ with the canonical grading and $(O(\mathbb{C}^n), |\cdot|_k)$, where

$$|f|_k = \sum_n |x_n| e^{k|\sigma(n)|}, \quad \forall f = \sum_s x_s z^{\sigma(s)} \in O(\mathbb{C}^n)$$

in view of the Cauchy estimates.

In this case tameness of S provides a positive integer A , such that for all $k = 1, 2, \dots$

$$\begin{aligned} \|S(f)\|_k &\prec \sum_s |x_s| e^{(k+A)|\sigma(s)|}, \\ \sum_s |x_s| e^{k|\sigma(s)|} &\prec \|S(f)\|_{k+A}, \end{aligned}$$

where as usual $\|f\|_k = \sup_{z \in D_k} |f(z)|$, and $D_k = \{z : \rho(z) \leq k\}$. Since the sequence $\{f_m\}$ constitutes a basis for $O(X)$, there is a $C_1 > 0$ and k_1 , so that

$$\sum_m |\beta_m| \|f_m\|_1 \prec \sum_m |\beta_m| e^{(1+A)|\sigma(m)|} \leq C_1 \left\| \sum_m \beta_m f_m \right\|_{k_1}$$

for every $f = \sum_m \beta_m f_m \in O(X)$. We choose, using regularity, polynomials p_m , $m = 1, 2, \dots$ so that

$$\|f_m - p_m\|_m \leq e^{|\sigma(m)|}, \quad m = 1, 2, \dots$$

and

$$\|f_m - p_m\|_{k_1} \leq \frac{1}{2C_1} \|f_m\|_1, \quad m = 1, 2, \dots$$

For $k > A + 1$ and $m \geq k$,

$$\|p_m\|_k \leq \|f_m\|_k + \|f_m - p_m\|_m \leq \|f_m\|_k + e^{|\sigma(m)|} \prec \|f_m\|_k.$$

Hence for every k large enough, there is a $c_k > 0$ such that

$$\|p_m\|_k \leq c_k \|f_m\|_k, \quad \forall m.$$

It follows that the operator Q defined by,

$$Q\left(\sum_m \beta_m f_m\right) = \sum_m \beta_m p_m$$

defines a continuous linear operator from $O(X)$ into itself. Moreover for a given $g = \sum_m \theta_m f_m$ in $O(X)$ and k large enough,

$$\begin{aligned} \|(Q - I)(g)\|_k &= \left\| (Q - I) \left(\sum_m \theta_m f_m \right) \right\|_k \leq \sum_m |\theta_m| \|f_m - p_m\|_k \leq \\ &\leq \sum_{m=1}^k |\theta_m| \|f_m - p_m\|_k + \sum_{n=k+1}^{\infty} |\theta_m| \|f_m - p_m\|_k \leq \\ &\leq \sup_{1 \leq m \leq k} \left(\frac{\|f_m - p_m\|_k}{\|f_m\|_1} \right) \sum_{m=1}^k |\theta_m| \|f_m\|_1 + \sum_{m=k+1}^{\infty} |\theta_m| e^{|\sigma(m)|} \prec \|g\|_{k_1}. \end{aligned}$$

In view of nuclearity of $O(X)$, the above estimates imply that $Q - I$ is a compact operator. In particular Q is Fredholm.

Now suppose there is an $f = \sum_m d_m f_m$, such that $Q(f) = 0$. We estimate:

$$\begin{aligned} \left\| \sum_m d_m f_m \right\|_k &= \left\| \sum_m d_m (f_m - p_m) \right\|_k \leq \sum_m |d_m| \|f_m - p_m\|_k \leq \\ &\leq \frac{1}{2C_1} \sum_m |d_m| \|f_m\|_k \leq \frac{1}{2} \left\| \sum_m d_m f_m \right\|_k. \end{aligned}$$

It follows that Q is one to one and hence an isomorphism. (see [10], p.671). Moreover we have:

$$\left\| Q \left(\sum_m d_m f_m \right) \right\|_k = \left\| \sum_m d_m p_m \right\|_k \leq \sum_m |d_m| \|f_m\|_k \prec \left\| \sum_m d_m f_m \right\|_{k+2A}.$$

We claim that Q is a tame isomorphism. In order to examine the continuity estimates of Q^{-1} we shall once again, turn our attention to the operator S . Consider the Hilbert scale $(H_t)_{t \geq 0}$,

$$H_t = \left\{ \xi = (\xi_m)_m : |\xi|_t = \left(\sum_m |\xi_m|^2 e^{2t|\sigma(m)|} \right)^{\frac{1}{2}} < \infty \right\}, \quad t \geq 0.$$

Fix a number A^- close to A yet $A^- < A$. The operator S , for large k induces maps:

$$\begin{aligned} H_k &\rightarrow O(D_{k-A^-}) \\ H_0 &\rightarrow O(D_0) \cap C(\overline{D_0}). \end{aligned}$$

In view of Zaharyuta interpolation theorem [37], S extends to be continuous from H_{tk} into $O(D_{t(k-A^-)})$ for each $0 \leq t \leq 1$. Similarly S^{-1} , for large k induces maps:

$$\begin{aligned} O(\overline{D_k}) &\rightarrow H_{k-A}, \\ O(\overline{D_0}) &\rightarrow H_0 \end{aligned}$$

and again by Zaharyuta interpolation theorem, S^{-1} extends to be continuous from $O(\overline{D_{tk}})$ into $H_{t(k-A)}$ for large k . Fix a large s and consider an $\overline{s} < s$ but near s . Choosing ks as large as needed, we see that S maps $O(\overline{D_{\overline{s}}})$ into $H_{\overline{s}}$ continuously and S^{-1} maps $H_{\overline{s}}$ into $O(D_{\overline{s}})$ continuously. In particular, the sequence $\{f_m\}$ forms a basis for the Fréchet space $O(D_s)$.

For $s \geq k_1$ arguing as above, we have,

$$\|p_m\|_s \leq \|f_m\|_s + \|p_m - f_m\|_s \prec \|f_m\|_s.$$

In particular these estimates and the fact that $\{f_m\}$ forms a basis for $O(D_s)$ allows us to conclude that for large s , the operator Q extends to a continuous operator from $O(D_s)$ into itself. The argument given above for the invertibility of Q on $O(X)$ applies for Q , this time as an operator from $O(D_s)$ into itself. This in turn will give us bounds on the continuity estimates of Q^{-1} . Namely for large k we have

$$\|Q^{-1}(f)\|_k \prec \|f\|_{k+1}.$$

Hence Q is a tame endomorphism of $(O(X), \rho)$. Now let $T = Q \circ S^{-1}$. This finishes the proof of the theorem. \square

REMARK 5.9. 1. The proof given above shows something more, namely that the polynomial basis found also constitute bases for the Fréchet spaces $O(\{z : \rho(z) < s\})$, for s large.

REMARK 5.10. 2. If we only assume that the Stein manifold X is S -parabolic then in view of Theorem 2.9 we can choose a Fréchet space isomorphism S from $O(\mathbb{C}^n)$, $n = \dim X$, onto $O(X)$. The general argument given in the first part of the proof of the above theorem is valid in this set up so as a corollary of the proof of the theorem we have:

COROLLARY 5.11. *Let (X, ρ) be a regular S -parabolic Stein manifold of dimension n . Then there exists an isomorphism from $O(\mathbb{C}^n)$ onto $O(X)$ that maps polynomials into ρ -polynomials. In particular $O(X)$ has a basis consisting of ρ -polynomials.*

References

- [1] Alexander H. Wermer J., On the approximation of singularity sets by analytic varieties II. Michigan Math J, 32 (1985), 227-235
- [2] Aytuna A. Krone J. Terzioglu T., Complemented infinite type power series subspaces of nuclear Fréchet spaces, Math. Ann. 283 no. 2 (1989), 193-202
- [3] Aytuna A., Krone J., Terzioglu T., Imbedding of Power Series Spaces and Spaces of Analytic Functions, Manuscripta mathematica 67 (1990), 125-142
- [4] Aytuna A., Algebraicity of Certain complex Analytic Subvarieties: Functional analysis point of view, Linear Topological Spaces and complex analysis, 1 TUBITAK Publications (1994), 1-13
- [5] Aytuna A., Linear tame extension operator from closed subvarieties, Proc. of the Amer. Math. Soc., 123:3 (1995), 759-763
- [6] Aytuna A., Sadullaev A., S -parabolic manifolds, TWMS, J.Pure Appl.Math., V.2, no.1, (2011), 249-253
- [7] Aytuna A., Sadullaev A., Parabolic Stein Manifolds, Mathematica Scandinavica, Vol. 114, no. 1 (2014), pp. 86-109
- [8] Berndtsson B., Ransford T.J., Analytic multifunctions, the equation and a proof of the Corona theorem, Pacific J. Math. 124 (1986), 57-72
- [9] Bremermann, H. J., On the conjecture of the equivalence of the plurisubharmonic functions and the Hartogs functions, Math. Ann., (1956), 131, 76-86
- [10] Edwards R.E., Functional Analysis: Theory and Applications, Dover publications – 1995
- [11] Federer H., Geometric measure theory, series Die Grundlehren der mathematischen Wissenschaften, Band 153, Springer-Verlag New York Inc., 1969
- [12] Fornaess J.E., Narasimhan R., The Levi problem on complex spaces with singularities, Math. Ann., V. 248, (1980), 47-72

- [13] Griffiths P., and King J., Nevanlinna theory and holomorphic mappings between algebraic varieties, *Acta mathematica*, V.130 (1973), 145-220
- [14] Klimek M., *Pluripotential Theory*, Clarendon Press, Oxford-New York-Tokyo, 1991
- [15] Meise R., Vogt D., *Introduction to Functional Analysis*, Clarendon Press, Oxford – 1997
- [16] Mitiagin B., Henkin G., *Linear Problems of Complex Analysis*, *Russ. Math. Surveys* 26, (1971), 99-164
- [17] Nishino T., Sur les ensembles pseudo concaves, *J. Math. Kyoto Univ.* 1-2 (1962), 225-245
- [18] Oka K., Note sur les familles de fonctions analytiques multiformes etc., *J. Sci. Hiroshima Univ.*, Ser. A, 4(1934), 93-98
- [19] Plesniak W., Remarques sur une généralisation de inégalité de Bernstein, *CR. Acad. Sc. Paris*, 284 (1977), 1211-1213
- [20] Poppenberg M., Tame subspaces of power series spaces, *Functional Analysis*, Trier– 1994, de Gruyter, Berlin – 1996, pp.365-375
- [21] Poppenberg M., Vogt D., *Tame splitting theory for Fréchet-Hilbert spaces*, *Functional Analysis: Lecture notes in pure and applied math.*, Marcel Dekker, New York, Basel, Hong Kong, 150, 1994
- [22] Rudin, W., A geometric criterion for algebraic varieties, *J. Math. Mech.*, V.17, (1968), 671-683.
- [23] Sadullaev A., Valiron's defect divisors for holomorphic map, *Mathem. Sbornik*, V. 108:4, (1979), 567-580. = *Math. USSR -Sb.* V.36, (1980), 535-547
- [24] Sadullaev A., Plurisubharmonic measure and capacity on complex manifolds, *Uspehi Math. Nauk*, Moscow, V.36 no.4(220), (1981) , 53-105 = *Russian Mathem. Surveys* V.36 (1981), 61-119
- [25] Sadullaev, A., Continuation plurisubharmonic functions from submanifold, *Dokl. Uzbek AN*, V.5, (1982), 3-4
- [26] Sadullaev A., Plurisubharmonic functions, *Encyclopedia of mathematical sciences*, VINITI, Springer-Verlag, V.8 (1985), 65-113 (59-106)
- [27] Sadullaev A., On analytic multifunctions, *Mathematical Notes*, V.83:5 (2008), 715-721
- [28] Sario, L.-M. Nakai, *Classification theory of Riemann surface*, Springer, Berlin-Heidelberg-New York – 1970, pp.216.
- [29] Shabat B.V., *Distribution of the value of holomorphic mappings*, Moscow, NAUKA – 1982; English transl.: *Transl. Math. Monogr.*, Providence – 1985
- [30] Slodkowski Z.: An analytic Set - Valued Selection and its applications to the Corona theorem, to Polynomial hulls and joint Spectra, *TAMS*, V.294, no.1, (1986), 367-378
- [31] Slodkowski Z., An open mapping theorem for analytic multifunctions, *Studia Math.* 122:2, (1997), 117-121
- [32] Stoll W., Value distribution of holomorphic maps into compact complex manifolds, *Lecture notes*, no. 135, Springer, Berlin-Heidelberg-New York – 1970
- [33] Stoll W., Value distribution on parabolic spaces, *Lecture notes*, no.600, Springer, Berlin-Heidelberg-New York – 1977
- [34] Vogt D., Charakterisierung der Unterräume von s , *Math. Z.* 155, (1977), no. 2, 109-111
- [35] Vogt, D., Charakterisierung der Quotientenräume von s und eine Vermutung von Martineau, *Studia Math.*, 67, (1980), no. 3, 225-240
- [36] Vogt, D., Power series space representations of nuclear Fréchet spaces, *TAMS*, 319, no.1, (1990), 191-208
- [37] Zaharyuta V.P. *Spaces of analytic Functions and Complex Potential theory*, *Linear Topological Spaces and Complex Analysis 1* TUBITAK Publications (1994), 74-146
- [38] Zeriahi A., Fonction de Green pluricomplexe a pole a l'infini sur un espace de Stein parabolique, *Math. Scand.*, V.69, (1991), 89-126
- [39] Zeriahi A., Approximation polynomial et extension holomorphe avec croissance sur une variété algébrique, *Ann. Polon. Math.* V.63:1, (1996), 35-50
- [40] Zeriahi A., A Criterion of Algebraicity for Lelong Classes and Analytic Sets, *Acta Math.*, 184(2000), 113-143

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